

Effective algorithm of analysis of integrability via the Ziglin's method.

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In this paper we continue the description of the possibilities to use numerical simulations for mathematically rigorous computer assisted analysis of integrability of dynamical systems. We sketch some of the algebraic methods of studying the integrability and present a constructive algorithm issued from the Ziglin's approach. We provide some examples of successful applications of the constructed algorithm to physical systems.

Keywords: Integrability, numerical approach, complexified systems, monodromy group, Ziglin's method.

I. INTRODUCTION

In this paper we describe the continuation of the research aimed at application of numerical methods to analysis of integrability of dynamical systems. In the first paper of the series ([1]) we have described the way of revealing the obstructions to real integrability in the *Liouville–Arnold sense* ([2]) by analyzing the topology of the phase space of the system. We have also discussed the possibilities of extending the approach to parametrized families of dynamical systems with the goal of searching for the regions of possible integrability.

The current paper is devoted to analysis of algebraic properties of integrable systems. We mention some recent results related to the study of complexified systems and systems of *variational equations* in particular. We pay special attention to the results of S.L. Ziglin ([3]) on the *meromorphic integrability*. We point out the major difficulties of application of these results to the study of integrability and propose an effective algorithm of a computer assisted construction significantly extending the range of their applicability. We give some examples of application of the method to dynamical systems having physical origin, and also mention some possible purely mathematical outcome.

II. ALGEBRAIC OBSTRUCTIONS TO INTEGRABILITY

Since for a given dynamical system there is no general approach for studying the existence of the sufficient number of arbitrary first integrals, a natural idea to restrict the class of first integrals comes out. A rather detailed review of the appropriate methods can be found in [4], some more recent methods are also well explained in [5]. Here we will only sketch some of them, that are important to understand our motivations and the results presented in this paper.

One can probably say, that the first algebraic method for analysis of integrability is the approach of S. Kovalevskaya developed in [6] by H. Yoshida for studying the polynomial first integrals of dynamical systems. Let us note that this restriction is rather reasonable since for mechanical systems with polynomial hamiltonian functions the natural integrals (energy, angular momentum etc.) are of this class. H. Yoshida applies this method to study the Euler equations as well as some hamiltonian systems having physical origin.

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A. Variational equations, monodromy group.

Some more recent methods of analysis of integrability deal with the complexified systems of differential equations. The key observation is that rather often for classical completely integrable systems the first integrals can be continued to the complex domain of the canonical variables depending on complex time remaining in the class of holomorphic or meromorphic functions. And branching of the solutions of the hamiltonian equations can create an obstruction to the existence of such first integrals. This idea permits in particular to find the relations between the parameters of a system necessary for complete integrability, that is select more regular systems from the family of a priori similar ones.

Following partially [5] let us formalize the above statement. The following approach actually dates back to A. Lyapunov who proposed to study the system of variational equations, it was developed by S. Ziglin in [3] who revealed the relation of the structure of the monodromy group to integrability.

Consider in \mathbb{C}^n (or any n -dimensional complex manifold) a system

$$\dot{x} = v(x), \quad (1)$$

where the dot denotes the derivative with respect to the complex time. Denote $x_0(t)$ – a particular solution of (1) and consider a small perturbation $x = x_0 + \xi$ of it. Note that here and in what follows the variables like x, v will be vector-valued (in \mathbb{C}^n , in a complex manifold or in the appropriate tangent bundle), and those like ξ can be identified to a respective tangent vector field in the whole phase space. Plugging x into (1) one obtains

$$\dot{\xi} = A(t)\xi + \dots, \quad A = \frac{\partial v}{\partial x}(x_0(t)).$$

Then the linearized system

$$\dot{\xi} = A(x_0(t))\xi \quad (2)$$

is called the system of *variational equations* along the particular solution $x_0(\cdot)$. We write $A(x_0(t))$ instead of just $A(t)$, to stress the fact that the matrix of this linear system depends on time only implicitly via its dependence on the particular solution. If the system (2) has multivalued solutions then the system (1) also does. We will be however interested in a more subtle property of these systems, namely branching of the solutions of the system of variational equations (2) when moving along the Riemann surface of the particular solution x_0 of (1).

Thus we will consider a system of n linear equations (2), where the entries of the matrix A are holomorphic functions defined in a connected neighbourhood of the Riemann surface x_0 . Locally for any given value of $\xi(t_0) = \xi_0$ there exists a unique holomorphic continuation of the solution (2). One can continue it along any path in x_0 but the result in the generic case need not be single-valued. Let γ be a loop (closed path) starting from $x(t_0)$ parametrized by some path on the complex plane. Constructing the continuation of the solution $\xi(t)$ of (2) defined in the neighbourhood of $x(t_0)$ along γ we obtain another solution $\xi_*(t)$ of (2) defined in the same neighbourhood. Since the system (2) is linear, there exists a complex $n \times n$ matrix T_γ such that $\xi_*(t) = T_\gamma \xi(t)$ for any ξ . Branching of the solutions of (2) corresponds to $T_\gamma \neq id$.

The set of all matrices $\mathcal{M} = \{T_\gamma\}$, corresponding to all the loops γ in x_0 , forms a group called the *monodromy group* of the linear system (2) along the solution x_0 . The matrices certainly depend on the choice of the base point on x_0 but all the groups \mathcal{M} are isomorphic. It is important to note that two homotopic loops produce the same monodromy matrices, therefore it is sufficient to consider only one representative γ from each homotopy class.

B. Results of S. Ziglin

Let $f(x)$ be a first integral of (1). One can write down the Taylor series for $f(x_0 + \xi)$

$$f(x_0 + \xi) = \sum_{m \geq 0} F_m|_{x_0}(\xi, t),$$

where F_m is the homogeneous form of degree m , single-valued on the Riemann surface x_0 . It is clear that a non-zero homogeneous form F_m of the lowest degree ($m \geq 1$) is the first integral of the system of variational equations (2), and thus invariant under the action of the monodromy group.

$$F_m(T\xi, t_0) = F_m(\xi, t_0), \quad T \in \mathcal{M}.$$

If $f(x)$ is meromorphic, it can be represented as the ratio of two holomorphic functions: $\frac{P(z)}{Q(z)}$. Then the analogous invariant is the ratio of the lowest degree forms corresponding to P and Q : $\frac{P_m(z)}{Q_k(z)}$. This is a strong condition on invariant functions, but even more important, the existence of such invariants imposes serious restrictions on the structure of the monodromy group.

Without going much into details let us just mention that for what follows it is important to exclude the cases when all the matrices of the monodromy group have trivial eigenvalues. This happens when the system possesses so-called symmetry fields; if this is the case one needs to perform the reduction of the system to normal variational equations. Let us also note that for the case of hamiltonian systems, which is of particular interest for us in the context of analysis of integrability, the elements of the monodromy group define symplectic affine transformations. In this case the eigenvalues of the monodromy matrices split into couples of mutually inverse complex numbers. The transformation is called *non-resonant* if its eigenvalues $(\lambda_1, \lambda_1^{-1}, \dots, \lambda_p, \lambda_p^{-1})$ satisfy the relation $\lambda_1^{k_1} \cdot \dots \cdot \lambda_p^{k_p} = 1$ if and only if all the k_i vanish.

Having defined the notion of the monodromy group we can state the following theorems:

Theorem 1. (Ziglin's lemma) *Let the monodromy group of the curve x_0 contain a non-resonant transformation g . Then the number of meromorphic first integrals of the hamiltonian equations in the connected neighbourhood of x_0 functionally independent from H is not bigger than the number of rational invariants of the monodromy group.*

Theorem 2. (S. Ziglin) *Let the monodromy group \mathcal{M} of the curve x_0 contain a non-resonant transformation g . If the hamiltonian equations admit $(n-1)$ meromorphic first integrals in the connected neighbourhood of x_0 functionally independent from H then any other transformation $g' \in \mathcal{M}$ preserves the fixed point of g and maps its eigendirections to eigendirections. If moreover no set of eigenvalues of g' forms a regular polygon on the complex plane (with the number of vertices ≥ 2), g' preserves the eigendirections of g (i.e. commutes with g).*

C. Differential Galois theory

Despite some important examples, the analytical computation of the monodromy group is a complicated task, solved only for some particular systems. In this context it is worth noting that there is an alternative approach proposed by J. Morales-Ruiz and J.-P. Ramis ([7, 8]) following the scheme of S. Ziglin. It establishes the relation between the properties of the differential Galois group and integrability. Not defining this group here (for the definition see e.g. [9]) let us only state the main result.

Theorem 3. (Morales-Ramis) *Let the hamiltonian system be completely integrable, then the identity component of the differential Galois group of the system of normal variational equations along any particular solution is abelian.*

When restricted to the case of complete integrability, i.e. the existence of a sufficient number of independent first integrals in involution, the theorem of Ziglin is the consequence of the one of Morales–Ramis. The latter is considered to be stronger ([5]) also in the sense that it permits to show non-integrability in some cases when the Ziglin's approach did not give any answer. The explanation of this fact is that for a fixed particular solution of (1) the differential Galois group contains the monodromy group, and therefore has more sources of non-commutativity. In this paper we however restrict our attention to the Ziglin's approach, leaving the results of Morales–Ramis to further studies.

III. EFFECTIVE ALGORITHM OF APPLICATION OF THE ZIGLIN'S METHOD.

We can formulate the standard way to analyze the integrability which is essentially based on the results reviewed in the previous section. The major steps are the following:

1. For a given complexified system, of differential equations (1) construct explicitly a solution x_0 , which as a function of complex time is viewed as a Riemann surface.
2. Write down the system of variational equations along x_0 . Perform a reduction of this system to the system of normal variational equations using known first integrals or symmetry fields.
3. Localize the singularities of the particular solution x_0 .
4. Construct the monodromy matrices, obtained by going along the loops around the singularities obtained in 3. – they generate the monodromy group.
5. Make a conclusion about the presence of the obstruction to integrability, based on the commutation relations between the matrices obtained in 4.

Let us comment on some subtleties of application of this method. The method is obviously not intended to prove integrability, and there are examples ([5, 8]) when the monodromy group is trivial, but the system is still non-integrable. But it can be also used to single out the relations between the parameters of the system, when integrability is possible. It is usually done, when the variational equations reduce to some “classical” well-studied systems. In principal, one can argue for integrability, when the monodromy group is commutative for *any* particular solution, but this is not worth the efforts, since knowing explicitly all the solutions of the initial system (1) one can perform a more detailed qualitative study of it. So the method can be developed mainly to search for the obstructions to integrability.

We can easily see two major difficulties of application of this method. First, one needs to construct an explicit particular solution of a system of differential equations. And second, this solution should be on one hand simple enough, so that the the monodromy group could be computed, and on the other hand, it should be non-trivial, so that the computed group could contain sufficient number of sources of non-commutativity. We employ numerical methods to overcome these difficulties, more precisely the idea is to compute numerically the generators of the monodromy group along a particular solution which is also obtained numerically.

For a given value of complex time t the value of the solution $x(t)$ can be computed numerically without any difficulty. One needs just to remember, that the obtained particular solution $x(t)$ should be considered not as a function of a point $t \in \mathbb{C}$ in a complex plane, but as a function of (the homotopy class of) a path going to t from the initial point t_0 . We can always write down the right hand sides of the variational equations $\dot{\xi} = A(x(t))\xi$ with the matrix A depending on an arbitrary particular solution x . The difficulty is that if we don't have the analytical expression of $x(t)$ we can not plug it into the variational equations, but with the above remark it does not make much sense in the general case. That is why we use a natural approach of solving the variational equations in parallel to the initial system along any path that interests us.

So, using the observations about the structure of the monodromy group from the previous section (II A) let us solve numerically the following system

$$\begin{aligned}\dot{x} &= v(x) \\ \dot{\Xi} &= A(x)\Xi,\end{aligned}\tag{3}$$

where the first line is identical to the initial complexified system of n differential equations (1), the second one is the matrix equation with A being the matrix of the system of variational equations depending explicitly on x , and Ξ – unknown $n \times n$ matrix. For initial data we take $x_0(t_0)$ – an arbitrary point in the phase space and $\Xi(t_0) = id_n$ – unit matrix. Going around the loop on the solution x we obtain in Ξ precisely the monodromy matrix corresponding to this loop. Thus going around all the singularities of the solution x we can construct the whole monodromy group.

It is important to understand that these loops are paths on the Riemann surface of the solution and not just on the complex plane. This is the major difficulty that one faces when applying the procedure: given the form of the equations (3) we are forced to integrate them against the complex time, but the object of interest for us is the behavior of the solution x . More precisely, when we go along a path in the complex plane and neither x nor Ξ have non-zero variation, this path does not produce a non-trivial monodromy generator. If Ξ has non-zero variation, one needs to check that the corresponding values of $x(t)$ have returned to the initial value. Only in this case Ξ is the generator of the monodromy group. If $x(t)$ has not returned to its initial value one needs to continue following the path. This difficulty is related to the fact that in general the solution x can not be parametrized by the complex time t , i.e. going around the loop in \mathbb{C} does not always produce a closed loop on x and the parametrization used in the original work of S. Ziglin should be considered as the parametrization of the loop on x . It is also clear that if x does not return to its

initial value after a finite number of going around a loop in \mathbb{C} (that is the topology of the singularity is logarithmic), such a loop does not correspond to any matrix in the monodromy group. But such an infinite branching by itself can be an obstruction to, say, analytic integrability, at least if the corresponding energy level is compact in the phase space. If all the branching points of x are of finite order (such a system satisfies the *generalized Painlevé property*), then the outcome of the procedure is the set of monodromy group generators. Having that, to prove non-integrability it is enough to find a couple of non-commuting matrices among them.

There is a couple of issues also worth being commented on. First, there is a difficulty which is more technical than conceptual, it is related to the localization of the singularities of the solution x (step 3 in the method above). Certainly it is impossible to go around all the loops in \mathbb{C} , so we have to restrict the analysis to some compact domain and go around the points of some finite grid. That is we do not try to construct the whole monodromy group, but only its subgroup, which is however usually enough to reveal the obstruction to integrability. Second, we have not discussed here the issue of resonant transformations. But, given the symplectic nature of the monodromy transformations this problem arises only from relatively large size of the system (1) in question. In this setting let us also note that one does not have to perform the reduction (in step 2 of the method) of the system of variational equations (2), it is only necessary to know that it can be performed. It is also important to mention that since the final action (step 5) is the verification an open condition of the commutator non-vanishing, it is perfectly correct from the point of view of the accuracy of numerical integration.

Summing up, let us formulate the **effective algorithm** of analysis of integrability via the Ziglin's method.

1. Write down (analytically) the system of equations (3) not fixing the particular solution.
2. Choose a bounded domain in \mathbb{C} and a finite grid of points in it with a distinguished point t_0 .
3. For each point choose a loop going around only it and starting at t_0 . Integrate numerically the system (3) along this loop taking $\Xi(t_0) = id$ as the initial conditions. Three cases are possible:
 - i. x and Ξ returned to initial values – this point gives a trivial transformation from the monodromy group.
 - ii. The value of x did not return to the initial values (within a given precision) – continue integrating around this loop. If x does not return to the initial values after a sufficiently large number of loops, one needs to analyze the density of the trajectory in the phase space.
 - iii. The values of x returned to the initial values after a finite number of loops, but of Ξ did not – store the matrix Ξ , it is one of the generators of the monodromy group.
4. Compute the pairwise commutators of all the matrices obtained in 3.iii. If there are non-vanishing commutators make a conclusion about non-integrability; if not choose another initial value of $x(t_0)$ in step 3.

IV. APPLICATION

In this section we apply the developed algorithm to some systems having mechanical interest. We start with the example that served one of the motivations to favor the approach of Morales–Ramis in comparison to the Ziglin's one – the system described by the Henon-Heiles hamiltonian. Within the framework of our algorithm we can also study the systems we were interested in while describing another approach in [1], namely the pendulum-type systems and satellite dynamics.

A. Henon-Heiles system

The Henon-Heiles hamiltonian describes a very simple model of a star moving close to the galactic center. It reads

$$H_h = \frac{1}{2}(p_1^2 + p_2^2) - q_2^2(A + q_1) - \frac{\lambda}{3}q_1^3.$$

In [5] (referring also to original works [7, 8]) the monodromy group and the differential Galois group were constructed for the hamiltonian equations governed by H_h using a rather simple particular solution satisfying $q_2 = p_2 \equiv 0$.

For this solution the monodromy group is trivial and does not obstruct integrability, while for $\lambda = 0$ the theory of Morales–Ramis permits to show that the system is non-integrable. But for $\lambda \neq 0$ and $A \neq 0$ one can only state non-integrability for $\frac{6}{\lambda} \neq \frac{k(k+1)}{2}$, $k \in \mathbb{Z}$. For $A = 0$ the approach does not give any result. Later Morales and Ramis performed a more detailed study, showing that the question of integrability is open only for $\lambda = 1, 2, 6, 16$. Using the fact that one is not forced to be restricted to the above particular solution in our numerical approach, we can study the remaining cases by our method using a more complicated one.

For example for $\lambda = 1$, $A = 0.25$ consider the initial data $(q_1, q_2, p_1, p_2) = (1, -0.4, -1.25, -0.3)$, $t_0 = 1$ (the figures are chosen arbitrarily not to obtain a vanishing solution for q_2, p_2). The commutator of matrices obtained by going along a loop around the points $(0.2 + 2.5i)$ and $(0.2 - 2.5i)$ is equal to

$$\begin{pmatrix} 0.390306 - 0.912711i & 0.657898 - 3.4626i & -1.09917 + 2.5655i & 0.65533 - 2.22299i \\ -0.936539 + 2.30099i & -0.636668 + 2.59892i & 1.31112 - 1.86606i & -1.72067 + 7.99614i \\ 0.314644 - 0.7603i & 0.273818 - 1.21218i & -0.51959 + 0.878226i & 0.575715 - 2.514i \\ 0.463812 - 1.0788i & 0.797168 - 4.19885i & -1.33512 + 3.13256i & 0.765952 - 2.56444i \end{pmatrix},$$

that results in non-integrability of the system.

B. Triple pendulum

A triple pendulum is a system of three mass-points (described by the radius-vectors \mathbf{r}_i) connected by weightless inextensible rods. We consider a free planar motion of this system which within the Lagrangian formalism can be described by a system with constraints of the form

$$\begin{aligned} \mathbf{r}_1^2 - l_1^2 &= 0, \\ (\mathbf{r}_i - \mathbf{r}_{i-1})^2 - l_i^2 &= 0, \quad i = 2, 3. \end{aligned}$$

We are not going to describe the formalism in full details here since we have already sketched it together with the motivations in [1] (cf. also references therein). Let us only recall that using a convenient parametrization by angles β_1, β_2 between the segments of the pendulum the system can be reduced by Routh transform to two degrees of freedom. One technical difficulty is that we need to have an explicit form of the system of variational equations, which is in this case much more complicated than in the previous example. Luckily, we can use algorithms of symbolic computation to obtain it; in this case we used the Sage ([10]) software package.

Turning to the results, for the initial data $(\beta_1, \dot{\beta}_1, \beta_2, \dot{\beta}_2) = (0.3, -1, -0.15, 0.5)$, $t_0 = 1$, the commutator of the matrices obtained by going along the loop around the points $(0.2 + 0.5i)$ and $(0.2 - 0.5i)$ (each of them six times), reads

$$\begin{pmatrix} 0.6170 - 0.6190i & 0.8124 + 0.8314i & -0.2373 - 0.3366i & -0.0180 + 0.3343i \\ -0.0658 - 0.1425i & 1.0420 + 0.0975i & -0.0447 + 0.1958i & -0.0189 - 0.3878i \\ 0.0206 - 1.0089i & 0.2544 + 1.5232i & 0.1464 + 0.5693i & 0.8635 - 1.4386i \\ -0.0047 + 0.0235i & -0.0163 - 0.0706i & -0.0227 - 0.0896i & 1.0258 + 0.1094i \end{pmatrix}.$$

That is the system is meromorphically non-integrable which is in perfect agreement with the results of [1].

C. Satellite dynamics

Let us now consider another example already mentioned in [1] – the motion of a dynamically symmetric satellite along a circular orbit ([11]). Using again the Routh reduction procedure one can describe the dynamics by the hamiltonian

$$H = \frac{p_\psi^2}{2 \sin^2 \theta} + \frac{p_\theta^2}{2} - p_\psi + \frac{1}{2} \sin^2 \psi \sin^2 \theta.$$

Consider the trajectory starting at $\psi = 0, \theta = 1, p_\psi = 0.1, p_\theta = 0, t = 0$ and continue it along the loop around the points $t = 4.8 + 0.8i$ and $t = 4.8 - 0.8i$ (two times around each of them). The respective commutator reads

$$\begin{pmatrix} 8849.8 + 13.2915i & 37.9467 - 126.071i & 2044.45 + 35.4031i & -1843.31 - 125.866i \\ -9456.28 - 239.498i & 311.834 - 62.6925i & -2350.67 - 37.0663i & 1972.34 + 197.277i \\ -34540.9 - 527.522i & 596.362 - 53.1627i & -8340.91 - 82.1358i & 7205.45 + 624.613i \\ 4032.64 - 556.427i & 1615.13 - 971.49i & 177.875 + 135.104i & -820.725 + 131.537i \end{pmatrix}.$$

That is the system is also meromorphically non-integrable. An interesting observation here is that the configuration of the system is parametrized periodically by the angles ψ, θ , that is we didn't have to make them return exactly to the initial condition, but only modulo 2π . Another interesting feature of the system revealed by the numerical experiment, is that the complexified dynamics of it is much more sophisticated than the real one. That is if the system is indeed locally integrable as we have conjectured in [1], these integrals can not be continued to meromorphic functions in the complex domain.

V. CONCLUSION

Thus, we have presented an algorithm for analysis of integrability of dynamical systems via the Ziglin's method using basically the properties of the monodromy group. As the examples show, it permits to extend the range of applicability of the method mainly because it resolves the problem of finding an explicit particular solution of the system. An important feature of the algorithm is that the trajectory obtained numerically, which is finally used for conclusion, is rather short and therefore can be computed with any given precision. It means that the algorithm indeed provides a rigorous method of computer assisted proof of non-integrability.

We have considered some mechanical examples that were not studied before, but the algorithm also has purely mathematical value. Namely, it is well adapted to computation of the monodromy group (or at least some subgroup of it) and also represents a serious step to analysis of the differential Galois group.

To conclude, let us mention that one of the motivations for developing constructive numerical methods of studying integrability for us is the qualitative analysis of the dynamical systems with delay or self-control appearing naturally in celestial mechanics and biological modeling.

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